A THIN SET OF LINES

BY

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ABSTRACT

Examples are given of functions $f(x)$ taking [0,1] into, or indeed onto, [0,1] in such a way that two dimensional measure of the set consisting of all points on all the straight line segments connecting $(x, 0)$ to $(f(x), 1)$ is zero.

The following question was posed by Dr. Belna of Michigan State University. Is there a function $f(\ )$ taking $[0,1]$ into $[0,1]$ such that, if $l(x)$ is the straight line segment connecting $(x,0)$ to $(f(x),1)$ in the plane, and L is the point set sum of all the points on all the line segments, i.e. $L = \bigcup_{0 \leqq x \leqq 1} \{(x,y):(x,y) \in l(x)\}$, the two dimensional Lebesgue measure of L is zero, i.e., $m_2(L) = 0$?

Two examples of such a function are given. Aspects of the problem are reminiscent of the Kakeya problem. The solution, however, is extremely simple and illustrates the connection of such problems to aspects of the Cantor set.

To illustrate a necessary property of such a function $f(\ )$, we make the negative observation.

If there exists a set A in $[0,1]$ of positive linear measure, such that the restriction of $f(\ )$ to A is monotone, then $m_2(L) > 0$.

It is sufficient to consider A closed and $f(x)$ continuous on A.

Let $(x_0, x_1), (x_2, x_3), \cdots, (x_{2n}, x_{2n+1}), \cdots$ be the intervals of the complement $\tilde{A}$ with respect to $[0,1]$ of A. Let T_i be the trapezoid (or triangle if $f(x_{2i}) = f(x_{2i+1})$ with vertices $(x_{2i}, 0)$, $(x_{2i+1}, 0)$ $(f(x_{2i}), 1)$,$(f(x_{2i+1}), 0)$. Then $T = \cup T_i$ contains the complement $\tilde{L}$ in the square (0,0), (01), (1,1), (1,0).

Since the T_i are disjoint, we have

$$m_2(T) = \sum_{i \geqq 0} m_2(T_i) = \frac{1}{2} \sum_{i \geqq 0} (x_{2i+1} - x_{2i}) + (f(x_{2i+1}) - f(x_{2i}))$$

$$= \frac{1}{2}[m(\tilde{A}) + m(f(\tilde{A})] = \frac{1}{2}[2 - m(A) - m(f(A))]$$

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where $m(\ )$ is linear Lebesgue measure.

So $m_2(\tilde{L}) < m_2(T) < 1$. Hence $m_2(L) > 0$.

Likewise if a measurable set B with $m_2(B) > 0$ exists for which $f(x) \leqq f(y)$ if $x > y$, for all pairs x, y in B. This may be seen by noting that $g_r(x) = rx + (1-r)f(x)$ is monotone upward for r sufficiently close to 1 and applying the argument used above. Thus the $f(x)$ in the examples will be seen to reorder all sets of positive measure.

EXAMPLE 1. We represent $x \in [0,1]$ in its triadic expansion $x = \sum_{i=0}^{\infty} x_i 3^{-i}$ where x_i takes values $0,1,2$. We take $f_i(x) = 2\delta(x_i, 2)$ and $v_i(x) = 2\delta(x_i, 1)$ where $\delta(a,b) = 1$ if $a = b$, 0 if $a \neq b$, and let $f(x) = \sum_{i=0}^{\infty} f_i(x) 3^{-i}$ and $v(x) = \sum_{i=0}^{\infty} v_i(x) 3^{-i}$. We note that $x = f(x) + v(x)/2$. The x coordinates of the intersection of L with $y = r$ will be the range, A_r, of the function

$$\begin{aligned} g_r(x) &= (1-r)x + rf(x) = (1-r)(f(x) + v(x)/2) + rf(x) \\ &= f(x) + (1-r)v(x)/2. \end{aligned}$$

The following figure illustrates the first two stages of the construction of L. Shaded areas are subsets of $\tilde{L}$.

That $m(A_r) = 0$ for all $r > 0$ implies that $m_2(L) = 0$ follows from Fubini's theorem.

We introduce the notation $C \otimes r = \{cr : c \in C\}$ $C \oplus a = \{c + a;\ c \in C\}$ and recall that for Lebesgue measure $m(\ )$ we have

$$m(C \otimes r) = rm(C), \quad m(C \oplus a) = m(C). \tag{1}$$

By construction it is clear that the range of $g_r(x)$ restricted to $\left[\frac{n}{3^k}, \frac{n+1}{3^k}\right]$ is of the form $\{A_r \otimes \left(\frac{1}{3}\right)^k \oplus d\}$ for a suitable d. In particular,

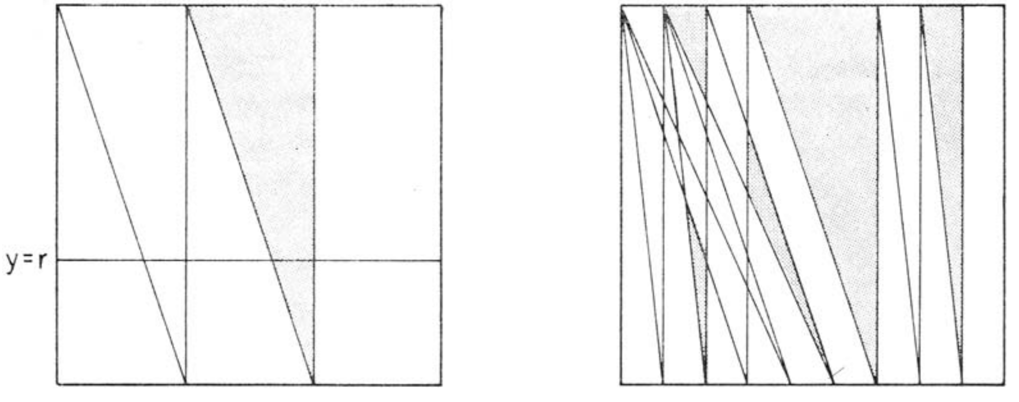

Fig. 1.

$$A_r = \left\{A_r \otimes \frac{1}{3}\right\} \bigcup \left\{A_r \otimes \frac{1}{3} \oplus \left(\frac{1}{3} - \frac{r}{3}\right)\right\} \bigcup \left\{A_r \otimes \frac{1}{3} \oplus \frac{2}{3}\right\}$$

By (1) we have $m(A_r) \leqq 3m(A_r \otimes \frac{1}{3})$, and, by induction, for every k,

$$m(A_r) \leqq 3^k m\left(A_r \otimes \left(\frac{1}{3}\right)^k\right). \tag{2}$$

By inspection, we have for the complement, $\tilde{A}$, of A,

$$\tilde{A}_r = \left\{\tilde{A}_r \otimes \frac{1}{3} \bigcap \left[0, \frac{1}{3} - \frac{r}{3}\right]\right\} \bigcup \left\{\left[\tilde{A}_r \otimes \frac{1}{3}\right] \bigcap \left]\tilde{A}_r \otimes \frac{1}{3} \oplus \left(\frac{1}{3} - \frac{r}{3}\right)\right] \bigcap \left[\frac{1}{3} - \frac{r}{3}, \frac{1}{3}\right]\right\}$$

$$\bigcup \left\{\tilde{A}_r \otimes \frac{1}{3} \oplus \left(\frac{1}{3} - \frac{r}{3}\right) \bigcap \left[\frac{1}{3}, \frac{2}{3} - \frac{r}{3}\right]\right\} \bigcup \left(\frac{2}{3} - \frac{r}{3}, \frac{2}{3}\right)$$

$$\bigcup \left\{\tilde{A}_r \otimes \frac{1}{3} \oplus \frac{2}{3}\right\}.$$

We use (1) to obtain

$$m(\tilde{A}_r) = \frac{1}{3} m(\tilde{A}_r) + \frac{1}{3} m(\tilde{A}_r) + \frac{1}{3} m(\tilde{A}_r) + \frac{r}{3}$$

$$- m\left\{\left[\tilde{A}_r \otimes \frac{1}{3}\right] \bigcap [\tilde{A}_r \otimes \frac{1}{3} \oplus \left(\frac{1}{3} - \frac{r}{3}\right) \bigcap \left[\left(\frac{1}{3} - \frac{r}{3}, \frac{1}{3}\right)\right]\right. .$$

Hence

$$\frac{r}{3} = m\left\{\left[\tilde{A}_r \otimes \frac{1}{3}\right] \bigcap \left[\tilde{A}_r \otimes \frac{1}{3} \oplus \left(\frac{1}{3} - \frac{r}{3}\right) \bigcap \left[\frac{1}{3} - \frac{r}{3}, \frac{1}{3}\right]\right]\right\}$$

$$\geqq \left\{\left[m\tilde{A}_r \otimes \frac{1}{3} \oplus \left(\frac{r}{3} - \frac{1}{3}\right)\right] \bigcap \left[\frac{1}{3} - \frac{r}{3}, \frac{1}{3}\right]\right\}.$$

It follows that

$$m\left\{A_r \otimes \frac{1}{3} \oplus \left(\frac{1}{3} - \frac{r}{3}\right) \bigcap \left[\frac{1}{3} - \frac{r}{3}, \frac{1}{3}\right]\right\} = 0$$

and so again by (1)

$$m\left[A_r \otimes \frac{1}{3} \bigcap \left(0, \frac{1}{3} - \frac{r}{3}\right)\right] = 0$$

But since, for some k, $\left(\frac{1}{3}\right)^k < \frac{1}{3} - \frac{r}{3}$, and, for $k \geqq 1$,

$$A_r \otimes \left(\frac{1}{3}\right)^k \subset A_r \otimes \left(\frac{1}{3}\right),$$

$$m\left\{A_r \otimes \left(\frac{1}{3}\right)\right\} \leqq m(A_r \otimes \frac{1}{3} \bigcap \left(0, \frac{1}{3} - \frac{r}{3}\right) = 0.$$

Taken with (2), this yields $m(A_r) = 0$.

A property of a subset of the cross product of the Cantor set is hereby shown to have a peculiar prismatic property. Let $K = \{(f, v)\}$ consist of the points of the cross product of the Cantor set with itself with the restriction that f_i and v_i are never simultaneously 2. The first two applications of this restriction are shown in the illustration below:

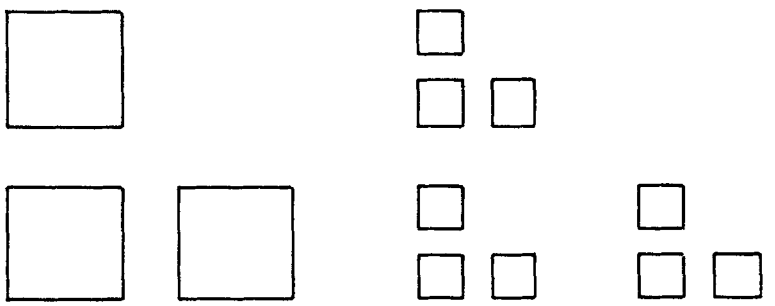

Fig. 2.

That $x = f(x) + v(x)/2$ states that projection of K in the direction $-\frac{1}{2}$ fills the interval, but that the projection of K by all other slopes between -1 and 0 are of measure zero is an interpretation of the example above. It happens that the only other directions where the projections fill an interval are -2 and 1.

EXAMPLE 2. We use a construction due to J. P. Kahane [2]. Let $E_{1/2} = \{x : x = 3\sum_{i=0}^{\infty} x_i 4^{-i}, x_i = 0, 1\}$, the Cantor set on $[0,1]$ obtained by leaving out middle halves. Let $L = \{(rx + (1-r)y, r) : x \in E_{1/2}, 0 \leqq r \leqq 1\}$ so that L is the point set sum of all segments connecting $(0, x)$ to $(1, y)$ for x and y taken from $E_{1/2}$. Using the notation of the previous example, $g_{1/3}(x) = \sum(x_i + 2y_i)4^{-i}$ and $g_{2/3}(x) = \sum(2x_i + y_i)4^{-i}$, so that $y = 1/3$ and $y = 2/3$ cut L in sets containing all points on $[0,1]$. Hence a segment in L connects each point of $(x, 1/3)$ to a point $(z, 1/3)$ where z may be found by replacing 1 for 2 and 2 for 1 in the base 4 expansion of x. Hence, if $m_2(L) = 0$, we have an example where each point of one segment is connected to some point of a parallel segment, in a one to one way with the area of the point set sum of the connected segments being zero.

We now prove that $m_2(L) = 0$. Since $x \in E_{1/2}$ implies that $x/4 \in E_{1/2}$ and $(x+3)/4 \in E_{1/2}$, we have $L = \bigcup_{i=1}^{4} L_i$ where $L_i = T_i L$; the T_i being the affine transformations with $y_i' = y$, all i, and $x_1' = x/4$, $x_2' = (x+3)/4$, $x_3' = (x + 3y)/4$, $x_4' = (x - 3 + 3y)/4$.

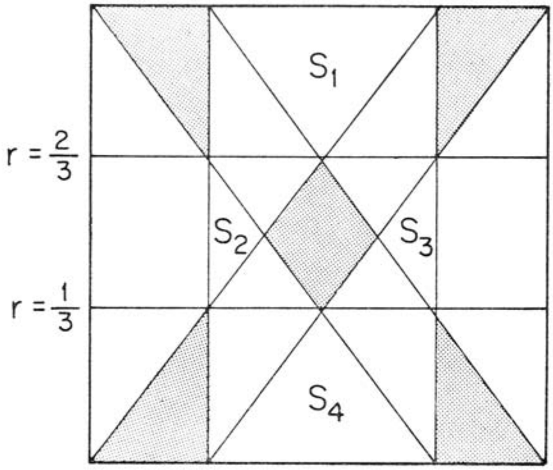

Fig. 3.

We note that $m_2\,(T_i A) = 1/4\, m_2\,(A)$. We denote by $c\,A$ the complement of A with respect to the unit square. We observe

$$m_2(c\,L) = \sum_{i=1}^{4} m_2(T_i c\,L) - \sum_{i \neq j} m_2(T_i c\,L \cap T_j c\,L) + \sum_{i=1}^{4} m_2(S_i)$$

wbere the intersections, and the triangles S_i are as indicated in Figure 3.

Since the sum of the areas of the intersections of the transforms of the unit square is the same as that of the areas of the S_i, it follows that the measure of the intersection of L with shaded regions is zero. By the Fubini theorem then, the

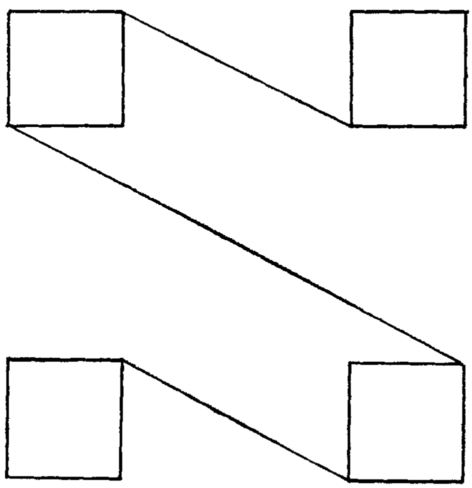

Fig. 4.

intersection of $y = r$ with L in the shaded area is of linear measure zero for almost all r. The procedure used in the previous example will then show that $m(L \cap (y = r)) = 0$ for all but the exceptional r, and hence, by another application of the Fubini theorem, $m_2(L) = 0$.

As before, this shows a prismatic property of $K = E_{1/2} \times E_{1/2}$.

The projection of K in the direction with slopes $1/2, 2, -1/2, -2$ fills an interval and almost all, (in fact the exceptional directions have slopes $\pm 2 \cdot 4^k$ with K *a* positive or negative integer) other directions the projections are of measure zero. This property was shown for a dense setof projections by Herzog and Piranian [1].

The proofs of a probabilist form used in [3] could have been used here, but a more arithmetic approach arizing from conversations with F. Herzog seemed more appropriate.

References

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3. J. R. Kinney, *A thin set of circles*, Amer. Math. Monthly **75** (1968), 1077–1081.